Beauducel, A. & Kersting, M. (2020). Identification of Facet Models by Means of Factor Rotation: A Simulation Study and Data Analysis of a Test for the Berlin Model of Intelligence Structure. *Educational and Psychological Measurement, 80,* 995-1019. https://doi.org/10.1177/0013164420909162

Educational and Psychological Measurement
2020, Vol. 80(5) 995–1019

DOI: 10.1177/0013164420909162
journals.sagepub.com/home/epm
SAGE

Identification of Facet Models by Means of Factor Rotation: A Simulation Study and Data Analysis of a Test for the Berlin Model of Intelligence Structure

André Beauducel[1] and Martin Kersting[2]

Abstract

We investigated by means of a simulation study how well methods for factor rotation can identify a two-facet simple structure. Samples were generated from orthogonal and oblique two-facet population factor models with 4 (2 factors per facet) to 12 factors (6 factors per facet). Samples drawn from orthogonal populations were submitted to factor analysis with subsequent Varimax, Equamax, Parsimax, Factor Parsimony, Tandem I, Tandem II, Infomax, and McCammon's minimum entropy rotation. Samples drawn from oblique populations were submitted to factor analysis with subsequent Geomin rotation and a Promax-based Tandem II rotation. As a benchmark, we investigated a target rotation of the sample loadings toward the corresponding faceted population loadings. The three conditions were sample size (n = 400, 1,000), number of factors (q = 4-12), and main loading size (l = .40, .50, .60). For less than six orthogonal factors Infomax and McCammon's minimum entropy rotation and for six and more factors Tandem II rotation yielded the highest congruence of sample loading matrices with faceted population loading matrices. For six and more oblique factors Geomin rotation and a Promax-based Tandem II rotation yielded the highest congruence with faceted population loadings. Analysis of data of 393 participants that performed a test for the Berlin Model of Intelligence Structure revealed that the faceted structure of this model could be identified by means of a

[1]University of Bonn, Bonn, Germany
[2]University of Gießen, Gießen, Germany

Corresponding Author:
André Beauducel, Institute of Psychology, Rheinische Friedrich-Wilhelms-Universität Bonn, Kaiser-Karl-Ring 9, Bonn 53111, Germany.
Email: beauducel@uni-bonn.de

Promax-based Tandem II rotation of task aggregates corresponding to the cross-products of the facets. Implications for the identification of faceted models by means of factor rotation are discussed.

Keywords

factor rotation, exploratory factor analysis, facet models, Berlin Model of Intelligence Structure

Consider a researcher who wants to investigate items by means of exploratory factor analysis (EFA). Although the researcher does not know exactly which factors will occur, she expects that there will be factors representing different measurement methods and factors representing different personality traits. Therefore, she expects that the factors are structured into a method facet (comprising a number of method factors, e.g., a factor for self-ratings and a factor for peer ratings) and a personality trait facet (comprising a number of trait factors). In consequence, each item measures one trait by means of one method and should therefore load on two factors. Thereby, the researcher follows the multitrait–multimethod approach (MTMM) of Campbell and Fiske (1959). The MTMM approach can be regarded as a specific form of a facet model comprising a method facet and a trait facet. ''A *facet* is a set that is a component of a cartesian product'' (Shye, 1998, p. 161; Guttman, 1954). For example, a facet A of factors a and b and a facet B of factors 1, 2, and 3 yields a cartesian product $A \times B = C$ with six elements (a1, a2, a3, b1, b2, b3). The items can be classified according to the elements of C so that each item has two substantial loadings, one loading on factor a or b and one loading on factor 1, 2, or 3. For example, there could be a method factor for all items from an interview, and another method factor for all items from a trait checklist, whereas a trait facet may comprise factors for the attitudes toward the father, superiors, and peers (see Campbell & Fiske, 1959, for a similar example). Since a facet contains a general attribute under which a number of factors can be subsumed (e.g., method in MTMM), the factors within one facet need not to be correlated. Campbell and Fiske (1959) provide several examples for models with factors that are structured into a method facet and a trait facet. It follows from a two-facet/MTMM model that each variable has salient loadings on two factors so that two parallel simple structures emerge (see Table 1). However, when EFA with subsequent rotation toward simple structure is performed for data that can be described by two parallel simple structures (e.g., one for the method factors and one for the traits), there is some risk that a researcher ignores that each item loads substantially on a method factor as well as on a trait factor. If only one salient loading of each variable on a factor is expected and if no rotation allowing for a faceted loading pattern is performed, two parallel simple structures may remain undetected, even when they are present in the data. However, if the data have such a complex structure, the identification of the two parallel simple structures of a two-facet model may be essential to

Table 1. A Perfect Two-Facet Model With Three Factors in Each Facet.

	Facet 1			Facet 2		
Variable	F1	F2	F3	F4	F5	F6
1	**.50**	.00	.00	**.50**	.00	.00
2	**.50**	.00	.00	.00	**.50**	.00
3	**.50**	.00	.00	.00	.00	**.50**
4	**.50**	.00	.00	**.50**	.00	.00
5	**.50**	.00	.00	.00	**.50**	.00
6	**.50**	.00	.00	.00	.00	**.50**
7	.00	**.50**	.00	**.50**	.00	.00
8	.00	**.50**	.00	.00	**.50**	.00
9	.00	**.50**	.00	.00	.00	**.50**
10	.00	**.50**	.00	**.50**	.00	.00
11	.00	**.50**	.00	.00	**.50**	.00
12	.00	**.50**	.00	.00	.00	**.50**
13	.00	.00	**.50**	**.50**	.00	.00
14	.00	.00	**.50**	.00	**.50**	.00
15	.00	.00	**.50**	.00	.00	**.50**
16	.00	.00	**.50**	**.50**	.00	.00
17	.00	.00	**.50**	.00	**.50**	.00
18	.00	.00	**.50**	.00	.00	**.50**

Note. Salient loadings are given in boldface. Facet 1 comprises factors F1, F2, and F3 and Facet 2 comprises factors F4, F5, and F6.

obtain an interpretable loading pattern. Basically, as for all factor analyzes, the factor rotation can improve the interpretation of the loading pattern while the overall amount of variance explained by the factors is not altered by rotation.

One example for factor models where two salient loadings of each variable on the factors are expected, is the bi-factor model (e.g., Holzinger & Swineford, 1937; Reise, 2012) also known as the nested-factors model (e.g., Mulaik & Quartetti, 1997). The bi-factor model can be conceived as a specific form of a two-facet model with only one factor in facet A (factor a) and more than one factor in facet B (e.g., factor 1, 2, and 3). In this context, the single factor in facet A can be regarded as a general factor, and the factors in facet B can be regarded as specific factors. An EFA with subsequent oblique rotation may reveal correlated first-order factors that might be entered into second-order EFA resulting in a second-order factor. Schmid and Leiman (1957) proposed a method that allows to transform such a higher order factor model into a constrained bi-factor model. Moreover, Jennrich and Bentler (2011) proposed a method that allows to perform EFA and a direct rotation toward a bi-factor solution. Since Jennrich and Bentler (2011) already demonstrated their method for bi-factor models, these models will not be further explored in the present study. Moreover, there are research contexts where multiple factors may occur in each facet so that a bi-factor model will not allow for an appropriate representation of the data.

Even then, a researcher might want to use EFA in order to explore a faceted structure. Therefore, we investigate whether two-facet structures based on multiple factors can be identified by means of EFA with subsequent factor rotation.

As will be shown in the following, up to now there has been no successful exploration of a priori unknown faceted structures by means of EFA of the measured variables (items or tasks). Relatedly, it should be noted that the utility of EFA for the investigation of faceted structures has been questioned and that smallest space analysis has been proposed as a method that may lead to a more parsimonious representation of the data (e.g., Guttman & Levy, 1991; Schlesinger & Guttman, 1969). These critical views might have discouraged a systematic investigation of the possibility to identify faceted structures by means of EFA with subsequent factor rotation. In spite of that, with the present study, we perform a systematic investigation of orthogonal and oblique two-facet structures by means of a simulation study to shed some light on this topic. An EFA approach to faceted structures is also of interest because the CFA approach has encountered problems with the identification of secondary loadings. Whereas fixing secondary loadings to zero is often unrealistic, the specification of a few freely estimated secondary loadings by means of successive model modifications also bears a number of problems (MacCallum et al., 1992). Exploratory structural equation modeling (ESEM) has been proposed to overcome these problems (Asparouhov & Muthén, 2009). Within the ESEM approach exploratory factor rotation is performed to estimate the measurement models. As faceted factor models may be incorporated into ESEM (e.g., within the MTMM approach), it is of interest how well available methods of factor rotation can identify faceted loading patterns.

Previous Approaches

Nevertheless, there have been attempts to use EFA in order to explore faceted structures. For example, Guilford (1967, 1975, 1988) tried to establish facets for the classification of intelligence factors (Operation, Content, Product) in the structure-of-intellect model. For each combination (cross-product) of Operations, Contents, and Products a factor was postulated. Guilford (1967) used EFA in combination with visual rotation and Procrustes/Target rotation to establish the factors postulated of the structure-of-intellect model. However, his way of providing evidence by means of factor rotation was not very compelling (Guttman & Levy, 1991; Süß & Beauducel, 2005). Another problem of Guilford's model was that the number of postulated factors was extremely large (more than 100), so that it was nearly impossible to demonstrate all these factors and their faceted structure with the limited number of variables/tasks per factor within a single EFA.

Another use of EFA to demonstrate a faceted structure of intelligence was performed by Jäger (1982, 1984), who proposed the Berlin Model of Intelligence Structure (BIS). Jäger started from EFAs of a very large set of intelligence tasks and identified a general intelligence factor, a facet comprising the four operation factors processing speed (S), memory (M), creativity (C), and reasoning/processing capacity

(R), and a facet comprising the three content factors figural intelligence (F), verbal intelligence (V), and numerical intelligence (N). Although the number of measured variables/tasks per factor was larger in Jäger (1982, 1984) than in Guilford (1967), the demonstration of a faceted intelligence structure by means of EFAs was only possible, when the measured variables/tasks were aggregated according to the operation facet or according to the content facet. Thus, for EFA of the operation facet, tasks were aggregated across content, resulting in task aggregates containing operation-homogeneous and content-heterogeneous tasks. For example, a processing speed aggregate comprised a verbal, a figural, and a numerical processing speed task. When three to five operation homogenous task aggregates per operation factor were formed, this resulted in 13 to 16 operation homogeneous aggregates. When these operation-homogeneous aggregates were entered into EFA, the four operation factors clearly emerged (Beauducel & Kersting, 2002; Jäger, 1982, 1984; Jäger et al., 1997). Similarly, when nine content homogenous task aggregates were entered into EFA, the three content factors could be demonstrated in these studies. Finally, when operation homogeneous and content homogeneous aggregates were entered simultaneously into EFA, the four operation factors and the three content factors were found (Jäger, 1982). Although the facet structure could clearly be found by means of EFAs of task aggregates, they were not based on EFAs of the tasks themselves. The use of task aggregates implies that some knowledge on the facets is already available so that the analysis is not purely exploratory, even when EFA is used. To sum up, Guilford (1967) used Target rotation and Jäger (1982) used a priori task aggregation for the demonstration of faceted structures. Thus, in the domain of intelligence structure, until now, faceted models have not been demonstrated by means of a purely exploratory and direct investigation of the measured variables by means of EFA.

In addition to EFA with subsequent Target rotation, EFA based on facet-homogeneous task aggregates, or smallest space analysis (Pfister & Jäger, 1992; Jäger et al., 1997), confirmatory factor analysis (CFA) has also been used for the investigation of faceted structures. Similarly, MTMM data have also been analyzed by means of CFA or alternative structural equation models (Marsh & Bailey, 1991). In the domain of intelligence research, CFA of facet-homogeneous aggregates has been used for the demonstration of the faceted structure of the BIS (Bucik & Neubauer, 1996; Süß et al., 2002). Moreover, 12 task aggregates corresponding to the cross-products of the four factors of the operation facet with the three factors of the content facet have been used in order to demonstrate simultaneously the operation factors and content factors of the BIS (Bucik & Neubauer, 1996; Süß & Beauducel, 2015). It should be noted that only the four operation factors could be shown by means of EFA of the 12 task aggregates corresponding to the cross-products (Bucik & Neubauer, 1996). Thus, the complete BIS facet structure could not be shown by means of an EFA of the 12 cross-product aggregates, whereas it could be shown by means of CFA. Since Bucik and Neubauer (1996) performed the EFA and the CFA for the same data set with the same 12 aggregates, the difference between the EFA- and CFA-results indicates that the problems with showing the complete facet structure by means of EFA could be due to methodological aspects related to EFA.

Further indications that the analysis of faceted structures could be a problem even for CFA can be found in Grayson and Marsh (1994), who investigated model identification issues of faceted CFA models from the perspective of the MTMM (Campbell & Fiske, 1959). According to the MTMM approach, a CFA should contain factors representing the relevant traits in one facet and factors for the measurement methods in another facet. Therefore, MTMM models can be regarded as a specific form of facet models (Süß & Beauducel, 2005), so that the results obtained algebraically by Grayson and Marsh (1994) for CFAs of MTMM data are also relevant for other faceted CFA models. Grayson and Marsh (1994) showed that the loading matrix is rank deficient for MTMM models. More specifically, they showed that the rank of an MTMM loading matrix equals the number of trait factors plus the number of method factors minus one. They also showed that the deficient rank of the loading matrix has consequences for model identification. For example, MTMM CFA models are not identified in a model where the correlations between all factors, are freely estimated. This model is referred to as the correlated traits and methods (CTM) model. Even when only within-trait factor correlations and only within-method factor correlations are freely estimated whereas all correlations of trait-factors with method-factors are fixed to zero, the MTMM CFA models are not identified. This model is the correlated traits, correlated methods (CTCM) model. However, when correlations between the trait-factors are freely estimated and when the intercorrelations between method factors as well as the correlations of trait factors with method factors are fixed to zero, the MTMM CFA model will typically be identified when there are at least three trait factors and three method factors. This model is termed the correlated traits, uncorrelated methods (CTUM) model.

To provide an identified MTMM CFA model, Eid (2000) proposed a model with correlated trait factors, correlated method factors, uncorrelated trait and method factors, and one method factor less than methods in the measured variables. Moreover, each variable of this model should be measured with only one method. The model is termed the correlated traits, correlated methods minus one (CTC(M-1)) model. This model implies that one method, is chosen as a comparison standard (Eid, 2000; Eid et al., 2003). For this reference method, no factor is defined so that the rank of the loading matrix corresponds to the result obtained by Grayson and Marsh (1994). Although CTC(M-1) models are probably well suited for the investigation of MTMM matrices (Eid, 2000), the idea that a one method factor is omitted as a reference standard, cannot easily be transferred to facet models outside the MTMM framework. Moreover, when the focus is on EFA, as in the present case, one would not know which factor should be omitted, because the factors are a priori unknown.

To summarize, it can be learned from CFAs of MTMM matrices that it is more easy to show facet models by means of CFA than by means of EFA and that the column rank of a two-facet loading matrix corresponds to the number of trait factors plus the number of method factors minus one and that this can lead to identification problems of CFA models that have been resolved in the MTMM context by means of specific modeling techniques (e.g., Eid et al., 2003).

An Approach for EFA

In the context of EFA, the rank deficiency of faceted loading structures (Eid, 2000; Grayson & Marsh, 1994) could lead to the problem that too many factors are retained for rotation of a faceted structure. That a two-facet loading matrix comprising q factors is based on $q-1$ unrotated, nonfaceted factors should be taken into account when EFA of faceted structures is intended. Accordingly, $q-1$ unrotated factors have to be extracted and rotated in order to identify q faceted factors. A four-step procedure is proposed here to identify q faceted factors for p variables by means of EFA:

1. Retain the unrotated loading matrix $\mathbf{L}_q$ from an extraction of q factors,
2. Retain the unrotated loading matrix $\mathbf{L}_{q-1}$ from an extraction of $q-1$ factors and concatenate $\mathbf{L}_{q-1}$ with a $p\times 1$ vector of zeroes resulting in $\mathbf{L}_{q-1,0}$.
3. Perform orthogonal Procrustes/target rotation (Schönemann, 1966) of $\mathbf{L}_{q-1,0}$ toward $\mathbf{L}_q$ as a target matrix, resulting in $\mathbf{L}_{q,q-1}$ a $p\times q$ loading matrix, but with rank $q-1$.
4. Perform exploratory factor rotation of $\mathbf{L}_{q,q-1}$.

Having q columns and rank $q-1$, $\mathbf{L}_{q,q-1}$ has the same properties as a faceted loading matrix so that it is possible to identify a faceted loading structure by means of an appropriate method of factor rotation.

In addition to the problem of the number of factors to extract, factor rotation may also be a problem for faceted structures. The original definition of simple structure, as it has been proposed by Thurstone (1947), allows for salient loadings of a variable on more than one factor. However, criteria that have been proposed for analytic rotation toward simple structure try to identify a perfect simple structure of minimal complexity (Browne, 2001), which is sometimes termed an independent clusters solution or a perfect cluster configuration. Thus, the focus of typical rotation criteria and methods of factor rotation is not on the identification of complex loading patterns as they occur in the context of faceted structures but on a perfect simple structure within one facet. It is therefore not surprising that simulation studies for the evaluation of methods of factor rotation were typically also based on simple structure models with salient loadings of each variable on only one factor (Velicer & Jackson, 1990). Although degraded simple structures with some substantial secondary loadings have sometimes been investigated (Schmitt & Sass, 2011; Weide & Beauducel, 2019) these degraded structures did not correspond to a faceted loading pattern. Since Guilford (1967) used target rotation, Jäger (1982) used a priori task aggregates, and simulation studies used only degraded simple structures one can conclude that population loading patterns representing more than one facet have not been explored systematically by means of EFA and subsequent (purely) explorative rotation toward simple structure.

The present study is an attempt to start to close this gap by means of a simulation study and an empirical study exploring the performance of available methods of rotation toward simple structure when a two-facet simple structure was to be expected. Although conventional rotation methods toward simple structure might not necessarily be suitable for the rotation of faceted structures, it can nevertheless not be excluded that some of them are more suitable than others. Moreover, the performance of at least some rotation methods is probably not really bad because a faceted structure implies that a perfect simple structure can be expected within each facet. Since a perfect population simple structure within each facet is expected, we expect that orthogonal methods of factor rotation toward simple structure as they have been described in Browne (2001) may be able to recover the faceted simple structure to some degree. However, Comrey's (1967) Tandem II rotation method deserves special attention here because it attempts to distribute the variance on as many factors as possible so that variables that are uncorrelated do not load on the same factor. As mentioned above, rotation toward a faceted loading matrix implies that a loading matrix with rank $q-1$ has to be rotated in order to represent q factors. Since the Tandem II rotation attempts to distribute the variance on the q factors it should be especially suitable for the identification of faceted loading structures from loading matrices with rank $q-1$.

To sum up, the aim of the present study is to explore which rotation methods are most suitable for the identification of a two-facet simple structure in the sample when a two-facet simple structure is given in the population. We consider a large number of rotation methods, orthogonal methods will be compared for orthogonal two-facet models and oblique rotation methods will be compared for oblique two-facet models. The focus will be on those rotation methods for which population two-facet models can be identified. Rather conventional orthogonal methods of factor rotation such as Varimax (Kaiser, 1958) and Equamax (Saunders, 1962) as well as more specialized methods such as Parsimax, Factor Parsimony (Crawford & Ferguson, 1970), Infomax (McKeon, 1968), and McCammon's (1966) minimum entropy rotation will be considered, but it is expected that Comrey's Tandem II rotation will perform quite well. Although it is not expected to be especially suitable for faceted structure, the Tandem I rotation will also be considered for orthogonal models in order to investigate Comrey's (1967) methods comprehensively. Oblique facet models for the population were considered for Oblimin rotation (with delta = 0; Jennrich & Sampson, 1966), for oblique versions of Equamax, Parsimax, and Factor Parsimony rotation, as well as for Geomin rotation (Yates, 1987). Since Tandem II rotation is a promising orthogonal method for the rotation toward facet structures, an oblique Tandem II rotation was performed by means of the procedure proposed by Hendrickson and White (1964) for Promax rotation. A target pattern based on the power of the Kaiser-normalized Tandem II loading pattern was used for oblique Target rotation. Thus, instead of using Varimax-rotation as prerotation method (as usual for Promax rotation) orthogonal Tandem II is proposed to be used as a prerotation method for further oblique rotation of correlated facet structures. In the following, this method will be denoted as Promax-based Tandem II rotation.

Method

The data generation and all data analyses were based on IBM SPSS Version 24. SPSS syntax examples for data generation and analysis of orthogonal and oblique models are given in the Supplementary Material (available online). For the simulation study population factor models based on an equal as well as a slightly unequal number of factors were investigated resulting in nine population models comprising 2 + 2, 2 + 3, 3 + 3, 3 + 4, 4 + 4, 4 + 5, 5 + 5, 5 + 6, and 6 + 6 factors so that the overall number of factors was $q = 4, \ldots, 12$. The size of the factor population main loadings was manipulated to be $l = .40$, .50, or .60. A perfect simple structure was realized within each facet. A perfect two-facet simple structure model based on $l = .50$ and 3 + 3 factors is given in Table 1. Orthogonal rotation methods were compared for two-facet population models based on uncorrelated factors, whereas oblique rotation methods were compared for population models based on factor intercorrelations of $r_f = .30$ between all factors. Since $r_f = .40$ would have resulted in a communality greater one for $l = .60$, a factor intercorrelation of $r_f = .30$ was investigated to use the same size of factor intercorrelations for all loading sizes of the oblique population models.

The sample sizes investigated were $n = 400$ or 1,000. This leads to 27 population loading matrices **L** (= 9 faceted models × 3 loading sizes) that were investigated for $r_f = .00$ and $r_f = .30$ leading to 108 conditions of the simulation study (= 27 population loading matrices × 2 levels of factor inter-correlations × 2 sample sizes). For each condition 1,000 samples were generated according to the following heuristic: For each participant q common factor scores $\mathbf{f_c}$ and p unique factor scores $\mathbf{f_u}$ were generated from normal distributions with $\mu = 0$ and $\sigma = 1$ by the method of Box and Muller (1958) from uniformly distributed numbers, which have been generated by the Mersenne twister integrated in SPSS. Observed variable scores $\mathbf{x}$ were generated from the factor scores by means of the factor model, with $\mathbf{x} = \mathbf{L}\mathbf{f_c} + diag(1\text{-}\mathbf{LL}')^{1/2}\mathbf{f_u}$. The observed variables of each sample were submitted to iterative principal axis factor analysis for the extraction of $q - 1$ factors and with subsequent rotation of q factors by means of the four-step procedure described above. For the population models based on uncorrelated factors, the following orthogonal rotation methods were compared: Varimax, Equamax, Parsimax, Factor Parsimony, Tandem I, Tandem II, Infomax, and McCammon's (minimum entropy) rotation. For the population models based on correlated factors, the following oblique rotation methods were compared: Oblique versions of Equamax, Parsimax, Factor Parsimony rotation, Oblimin, Geomin, and a version of Promax rotation (Hendrickson & White, 1964) based on orthogonal Tandem II prerotation. The rotation methods were based on Bernaards and Jennrich's (2005) gradient projection algorithm (see Jennrich, 2001). As a benchmark, orthogonal target rotation (Schönemann, 1966) toward the faceted population loadings was investigated for the population models based on uncorrelated factors, and oblique target rotation was investigated for the population models based on correlated factors. The dependent measure was the coefficient of congruence (Tucker, 1951) of the rotated sample loading matrices with the respective population loading matrix with a perfect two-facet simple structure. The mean coefficient of congruence M(c) was computed as an average of the congruence coefficients of the sample loadings of each factor with the corresponding population factor loadings. Since the population models represent perfect faceted simple structures without any distortion that might occur due to measurement error, a rotation method that is able to identify faceted loading patterns in the sample should have a large M(c). Therefore, M(c) $>$ .98 was used as a criterion for a very good identification of the faceted loading patterns.

Two additional analyses were reported for the l = .50 (n = 400, 1,000) condition of the orthogonal population models to investigate the robustness of the results of the overall simulation study. It has been shown that a number of different starting solutions might improve the performance of the gradient projection algorithm for factor rotation (Weide & Beauducel, 2019). As random starts need a substantial amount of computation time, an additional simulation was performed with only 200 runs for 10 conditions: The orthogonal population models with 2 + 2, 3 + 3, 4 + 4, 5 + 5, and 6 + 6 factors (= 5 population models $\times$ 2 sample sizes) to compare the sample average M(c) for 20 random start solutions with the sample average M(c) reached with a single start solution in the overall simulation. Moreover, the root mean squared error (RMSE) was computed for each orthogonal loading matrix as the root mean squared difference between the sample and orthogonal population loading matrices to investigate whether the results are similar to those found with the coefficient of congruence.

Results

Simulation Results for Orthogonal Population Models

The congruences (averaged across the factors of a model) of rotated population loadings with the two-facet population model with l = .40 are reported in Figure 1 (A and B). For Equamax, Parsimax, Factor Parsimony, and Tandem II rotation of the population loading matrices, a nearly perfect congruence with the population two-facet models occurred for q = 8 factors. The congruence of the rotated population loading matrices with the two-facet population loading matrices increased with q for these methods (Figure 1A). For Varimax, Infomax, Tandem I, and McCammon rotation the congruence of the population rotated loading matrices with the population two-facet models did not show a systematic pattern (Figure 1B). Whereas the congruence was generally low for Varimax, Tandem I, and McCammon's rotation, some nearly perfect congruences occurred for Infomax rotation and one nearly perfect congruence

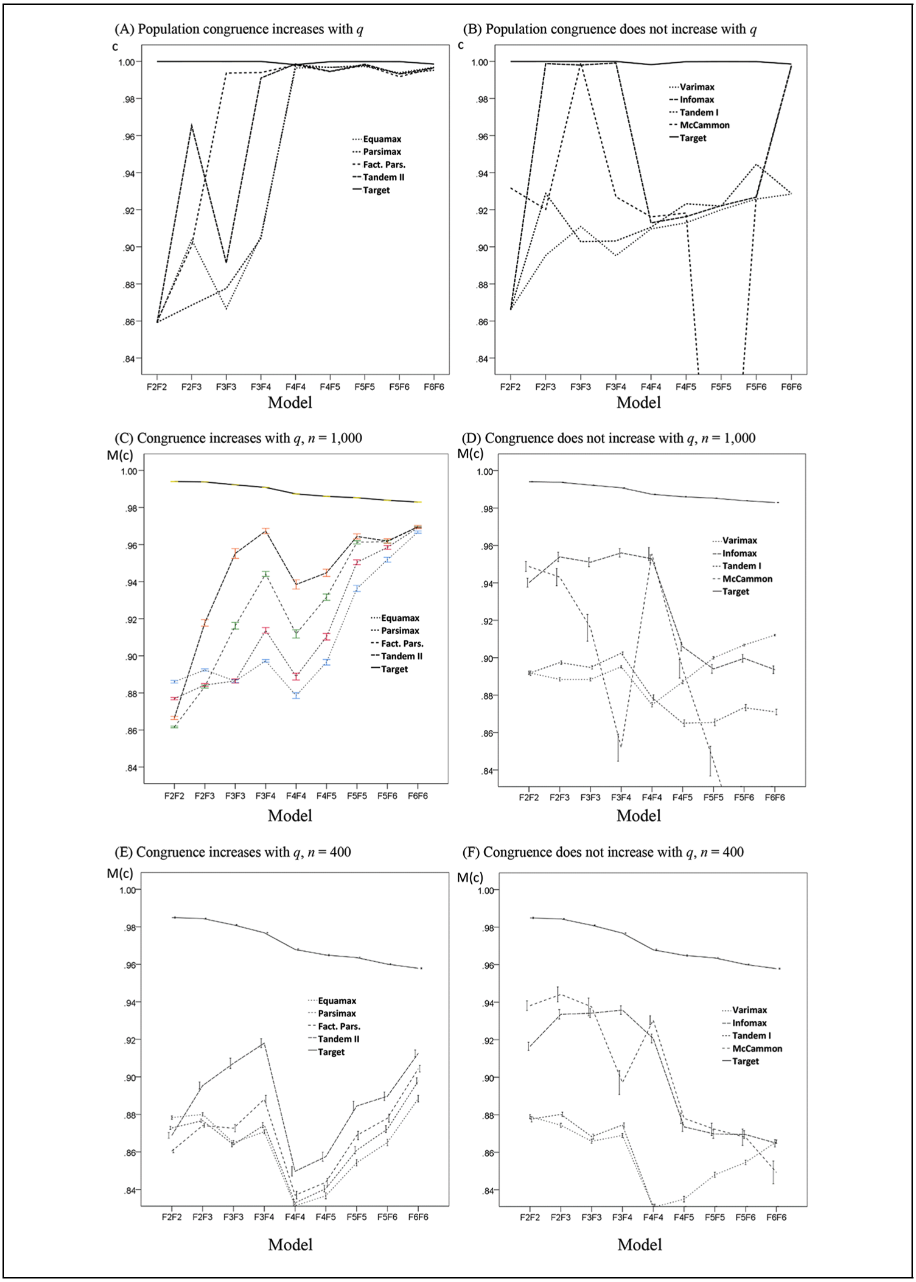

Figure 1. Population congruence (c) in (A) and (B), sample average of *M*(c) for orthogonal loading matrices with orthogonal population two-facet simple structure for *l* = .40 in (C) to (F); the model axis gives the number of factors for each of the two facets; the error bars mark the 95% confidence interval.

occurred for McCammon's rotation. Although M(c), the average congruence of rotated sample loadings with the corresponding two-facet population loadings, was generally lower than the congruences of the rotated population loadings with the two-facet population loadings, the pattern of results was similar (Figure 1C-F). M(c) for orthogonal Target rotation of the sample loadings toward the population loadings is reported as a reference line. An increase of M(c) with q was observed for Equamax, Parsimax, Factor Parsimony, and Tandem II (Figure 1C and E). Such a tendency was not observed for Varimax, Tandem I, Infomax, and McCammon's minimum entropy (Figure 1D and F). It should be noted that M(c) was greater than .98 only for the Target rotation for n = 1,000 and that for n = 400 and $q > 5$ even Target rotation did not reach an M(c) greater than .98. It can be concluded that for l = .40 a two-facet orthogonal simple structure cannot easily be identified with EFA.

For l = .50, the pattern of congruences of rotated population loadings with population two-facet models was similar to the pattern for l = .40 (Figure 2A and B). Whereas a nearly perfect congruence was found for $q \geq 8$ for Equamax, Parsimax, Factor Parsimony, and Tandem II rotation (Figure 2A), a rather unsystematic pattern occurred for Infomax and McCammon's rotation while Varimax and Tandem I rotation yielded generally low congruences (Figure 2B). This pattern of results was also found for the M(c) average across samples (Figure 2C-F). For n = 1,000, the sample average of M(c) is greater than .98 for Tandem II rotation when $q \geq 6$ (with three and more factors in each facet) and Equamax, Parsimax, and Factor Parsimony reach an average M(c) greater than .98 for $q \geq 10$ (see Figure 2C). There is no rotation method with an average M(c) greater than .98 for $q < 6$, n = 1,000, but it should be noted that Infomax rotation had the largest average M(c) for these small population models (see Figure 2D). For l = .50 and n = 400 only the Target rotation reached an average M(c) greater than .98, but Tandem II rotation yielded still the largest average M(c) for $q \geq 6$, and Infomax rotation yielded the largest average M(c) for $q < 6$. As an example, the sample average Tandem II rotated loadings for q = 6, l = .50 and n = 400 are given in Table 2. Even when M(c) is slightly below .98 for these models (see Figure 2E), the averages of the loadings are close to the population loadings and the standard deviations of the loadings are $\leq$ .10. To sum up, for l = .50, n = 400 and $q \geq 6$ a satisfying exploratory rotation of two-facet orthogonal loading patterns by means of Tandem II rotation is possible, and with n = 1,000 it is rather likely to reveal a $q \geq 6$ two-facet model by means of Tandem II rotation. However, small sample sizes could substantially impair the possibility to identify faceted loading patterns for l = .50.

For l = .60, the congruence of rotated population loadings with population two-facet loadings was nearly perfect for Tandem II and Factor Parsimony rotation for $q \geq 6$ and it was nearly perfect for Parsimax and Equamax rotation for $q \geq 7$ (Figure 3A). No systematic increase of congruences with q occurred for the remaining methods (Figure 3B). Tandem II rotation reached a sample average M(c) greater

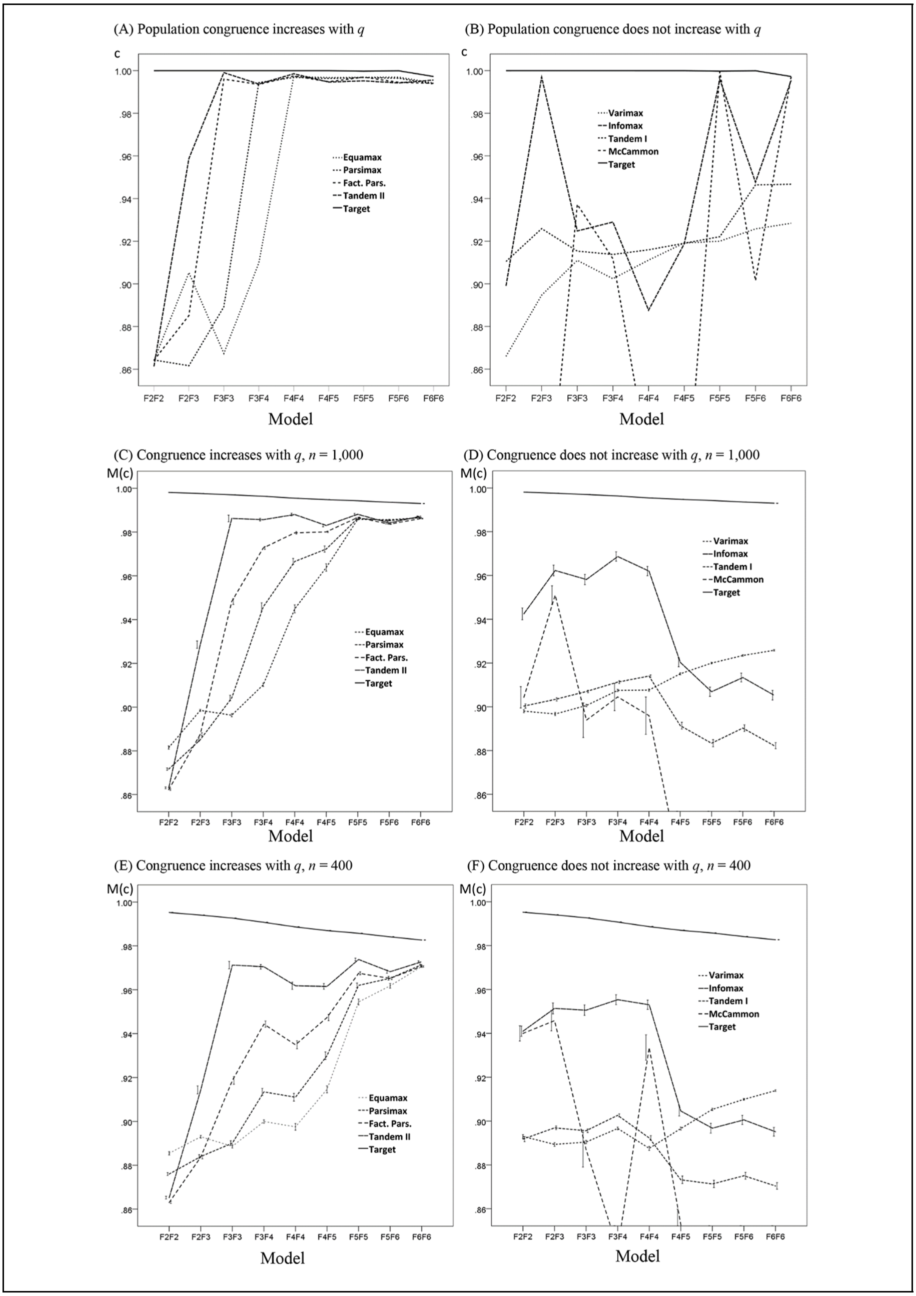

Figure 2. Population congruence (c) in (A) and (B), sample average of M(c) for orthogonal loading matrices with orthogonal population two-facet simple structure for l = .50 in (C) to (F); the model axis gives the number of factors for each of the two facets; the error bars mark the 95% confidence interval.

Table 2. Mean (*SD*) of Tandem II Rotated Loadings for 1,000 Samples With *n* = 400 Drawn From a Population Model With Two-Facets and Three Factors in Each Facet.

	Facet 1			Facet 2		
Variable	F1	F2	F3	F4	F5	F6
1	**.49** (.07)	.01 (.07)	.00 (.07)	**.48** (.10)	.00 (.07)	.00 (.07)
2	**.49** (.08)	.00 (.06)	.00 (.07)	.00 (.08)	**.48** (.09)	.00 (.08)
3	**.49** (.07)	.00 (.07)	.00 (.07)	.00 (.08)	.00 (.08)	**.48**(.09)
4	**.49** (.08)	.01 (.07)	.00 (.07)	**.48** (.10)	.01 (.07)	.00(.07)
5	**.49** (.07)	.00 (.06)	.00 (.07)	.00 (.07)	**.48** (.09)	.00(.08)
6	**.49** (.07)	.00 (.07)	.00 (.07)	.00 (.08)	.00 (.08)	**.48** (.09)
7	.00 (.07)	**.49** (.07)	.00 (.07)	**.48** (.09)	.00 (.08)	.00 (.07)
8	.00 (.07)	**.49** (.07)	.01 (.07)	.00 (.08)	**.48** (.09)	.00 (.08)
9	.00 (.07)	**.49** (.08)	.00 (.07)	.00 (.07)	.00 (.08)	**.48** (.09)
10	.00 (.07)	**.49** (.07)	.00 (.06)	**.48** (.09)	.01 (.07)	.00 (.07)
11	.00 (.07)	**.49** (.07)	.00 (.07)	.00 (.07)	**.48** (.09)	.00 (.08)
12	.00 (.07)	**.49** (.07)	.00 (.07)	.00 (.07)	.01 (.08)	**.48** (.09)
13	.01 (.07)	.00 (.07)	**.49** (.07)	**.48** (.10)	.00 (.07)	.00 (.08)
14	.00 (.07)	.00 (.06)	**.49** (.08)	.00 (.08)	**.48** (.08)	.00 (.08)
15	.00 (.07)	.00 (.07)	**.49** (.07)	.00 (.07)	.00 (.07)	**.48** (.10)
16	.00 (.07)	.00 (.07)	**.49** (.07)	**.48** (.10)	.00 (.07)	.00 (.08)
17	.00 (.07)	.00 (.06)	**.49** (.07)	.00 (.08)	**.49** (.08)	.00 (.08)
18	.00 (.07)	.00 (.07)	**.49** (.07)	.00 (.08)	.00 (.07)	**.48** (.10)

Note. Mean loadings $>.30$ are given in boldface. Facet 1 comprises Factors F1, F2, and F3 and Facet 2 comprises Factors F4, F5, and F6.

than .98 for $q \geq 6$ for n = 1,000 (Figure 3A) as well as for n = 400 (Figure 3C). Equamax rotation reached an average M(c) greater than .98 for $q \geq 7$ for n = 1,000 (Figure 3A) and for $q \geq 8$ for n = 400 (Figure 3C). Infomax rotation had the largest average M(c) for $q < 6$; however, the average M(c) does not reach .98 (see Figure 3B, 3D). Thus, for $q \geq 6$, $l \geq .60$, and $n \geq 400$, there is a good chance that an orthogonal two-facet loading pattern (i.e., given in the population) can be identified in the sample by means of Tandem II rotation. For $q < 6$, $l \geq .60$, and $n \geq 400$ one may try out Infomax rotation, but the chance to identify a given orthogonal two-facet loading pattern in the sample is not very high.

An investigation of the effect of the number of random starts was also performed. For n = 1,000, the random starts reduced the sample average M(c) for Equamax, Parsimax, and Factor Parsimony rotation, but they did not alter the sample average M(c) for Tandem II rotation (see Supplemental Figure S1A and B). Moreover, the random starts considerably improved M(c) for McCammon's rotation although M(c) was not very high even after the improvement (Supplemental Figure S1C and D). Although the random starts only slightly improved M(c) for Infomax rotation, the

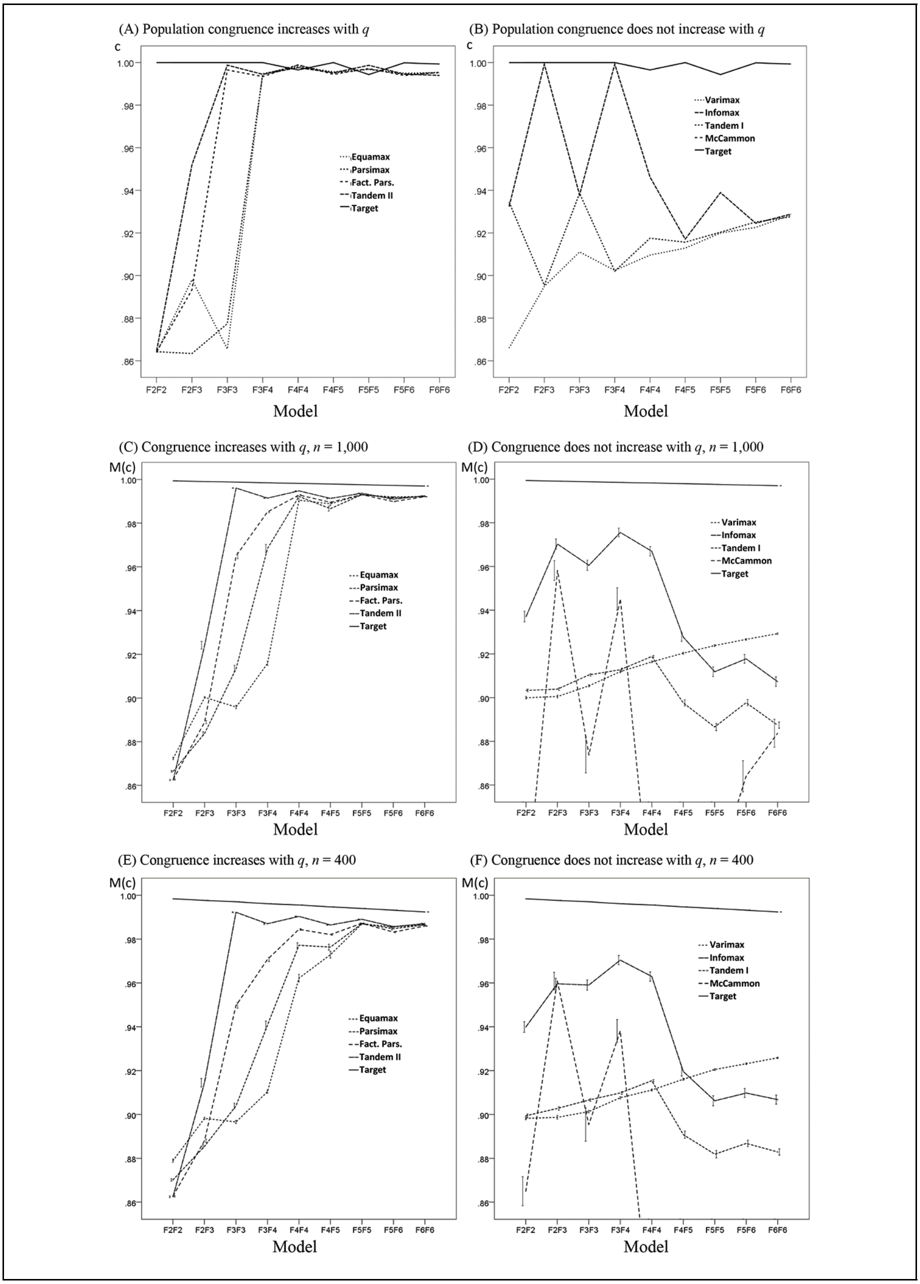

Figure 3. Population congruence (c) in (A) and (B), sample average of *M*(c) for orthogonal loading matrices with orthogonal population two-facet simple structure for *l* = .60 in (C) to (F); the model axis gives the number of factors for each of the two facets; the error bars mark the 95% confidence interval. The population congruences for McCammon's rotation are all $<.86$.

resulting M(c) was close to .96 for n = 1,000 and $q \leq 10$, indicating that Infomax could be an appropriate method with a larger number of random starts. Again, for $q < 6$ Infomax outperforms Tandem II rotation (Supplemental Figure S1E and F). For n = 400 the random starts tend to reduce the sample average M(c) for all methods except McCammon's and Infomax rotation, which outperforms Tandem II rotation for $q < 8$ (Supplemental Figure S1G and H).

The analysis for the sample averaged RMSE yielded similar results as M(c) in that Tandem II rotation yielded the smallest RMSE for $q \geq 6$ (see Supplemental Figure S2A and C). However, McCammon's minimum entropy and not Infomax rotation yielded the smallest RMSE for $q < 6$, although the difference was small (Supplemental Figure S2B and D). This indicates that the results obtained for M(c) with Tandem II rotation were robust but that Infomax rotation or McCammon's rotation might be considered for $q < 6$. To summarize, the results of the simulation study for orthogonal models reveal that Tandem II, Infomax, and McCammon's rotation could reveal faceted EFA loading patterns and that, overall, Tandem II rotation yielded the highest congruence of rotated loadings with orthogonal population two-facet models.

Simulation Results for Oblique Population Models

Overall, the congruence of rotated population loadings with corresponding oblique two-facet population loadings was not large for oblique Equamax, Parsimax, and Factor Parsimony rotation (Figures 4B, 5B, and 6B). However, Geomin (with ε = 0.1) and Promax-based Tandem II rotation (with Power = 2) of population loadings resulted in high congruences with population loadings for l = .40 (Figure 4A) and l = .50 (Figure 5A) and in nearly perfect congruence for l = .60 and $q \geq 6$ (Figure 6A). It was therefore decided to perform the sample-based simulations only for Geomin and Promax-based Tandem II rotation.

M(c) for averaged rotated sample loadings with two-facet population loadings was below .90 for $q > 7$ and l = .40 (Figure 4C and 4D), greater than .90 for l = .50, $q \leq 11$, and n = 1,000 (Figure 5C) and about .95 for l = .60, $q > 6$, and n = 1,000 (Figure 6C). M(c) was larger for Geomin rotation than for Promax-based Tandem II rotation. The effect of sample size was substantial for l = .40 (Figure 4) and was rather small for larger salient loadings. Overall, the congruence of rotated sample loadings with two-facet population loadings was smaller for oblique models than for orthogonal models.

Empirical Study

A convenience sample of 393 voluntary German participants (159 females; age in years: M = 15.38, SD = 0.89) worked on the BIS-4 test (Jäger et al., 1997) comprising 45 tasks representing the two-facet structure of the BIS. Participants provided

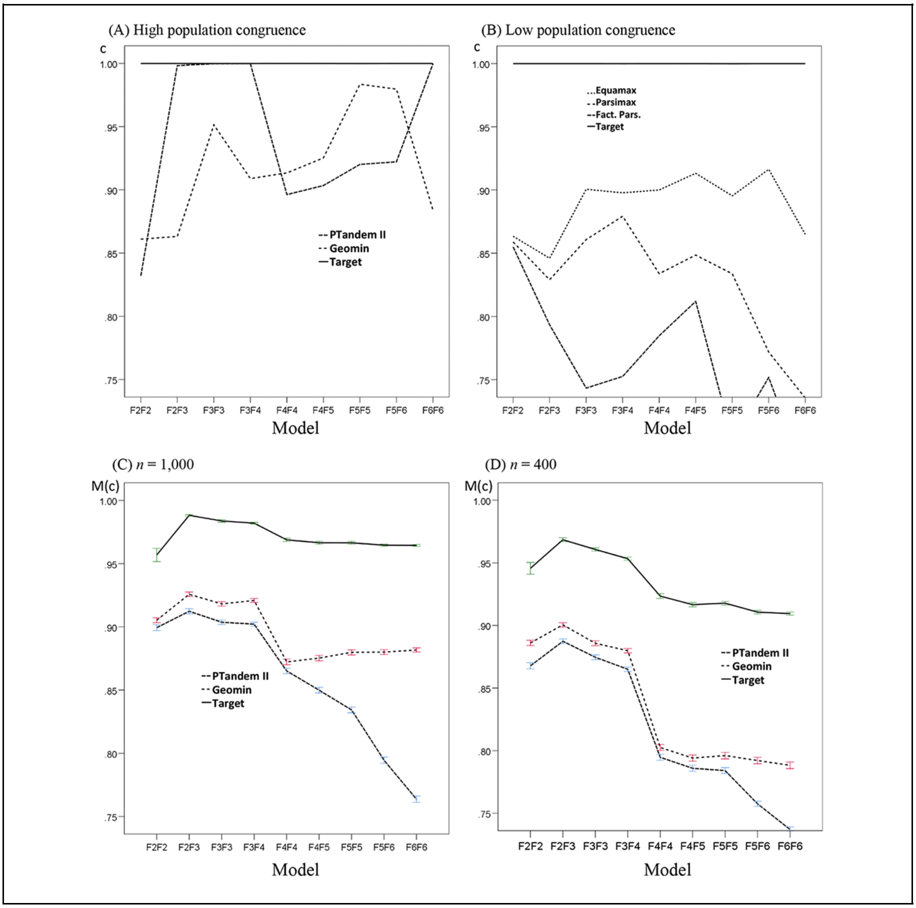

Figure 4. Population congruence (c) in (A) and (B) and sample average of *M*(c) in (C) and (D) for oblique loading matrices with oblique population two-facet simple structure for l = .40, factor-intercorrelation r_f = .30; the model axis gives the number of factors for each of the two facets; the error bars mark the 95% confidence interval. "PTandem II" denotes orthogonal Tandem II pre-rotation followed by Promax (Power = 2) rotation.

verbal informed consent. The study was conducted in accordance with the World Medical Association's Declaration of Helsinki and approved by the ethics board of the Institute of Psychology (Freie Universität Berlin, Germany). The intercorrelation of the tasks and the classification of the tasks according to the facets of the BIS is shown in Supplemental Table S1. As the simulation study revealed that Tandem II rotation is promising for orthogonal rotation of two-facet models and that Promax-

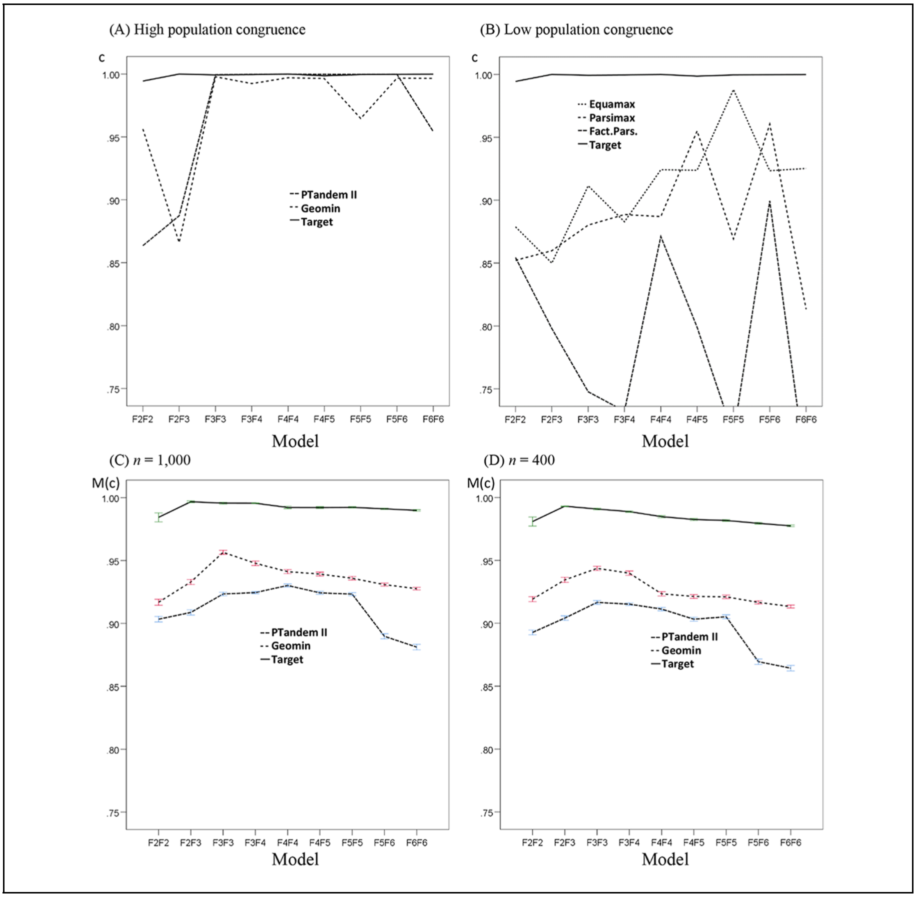

Figure 5. Population congruence (c) in (A) and (B) and sample average of *M*(c) in (C) and (D) for oblique loading matrices with oblique population two-facet simple structure for l = .50, factor-intercorrelation r_f = .30; the model axis gives the number of factors for each of the two facets; the error bars mark the 95% confidence interval. "PTandem II" denotes orthogonal Tandem II pre-rotation followed by Promax (Power = 2) rotation.

based Tandem II rotation can be performed to find oblique two-facet models, we performed Tandem II rotation and a Promax-based Tandem II rotation based on an EFA of the 12 task aggregates corresponding to the cross-products of the BIS factors. This allows to directly compare the orthogonal Tandem II solution with the corresponding Promax-based oblique solution, so that the comparison of the orthogonal and oblique solution is not confounded with a change of the criterion for optimal rotation. The intercorrelation of the 12 task aggregates is given in the Supplemental Table S2. First, $q - 1 = 6$ factors were extracted and $q = 7$ factors were rotated according to the four-step procedure described in the introduction. The congruence of the Tandem II

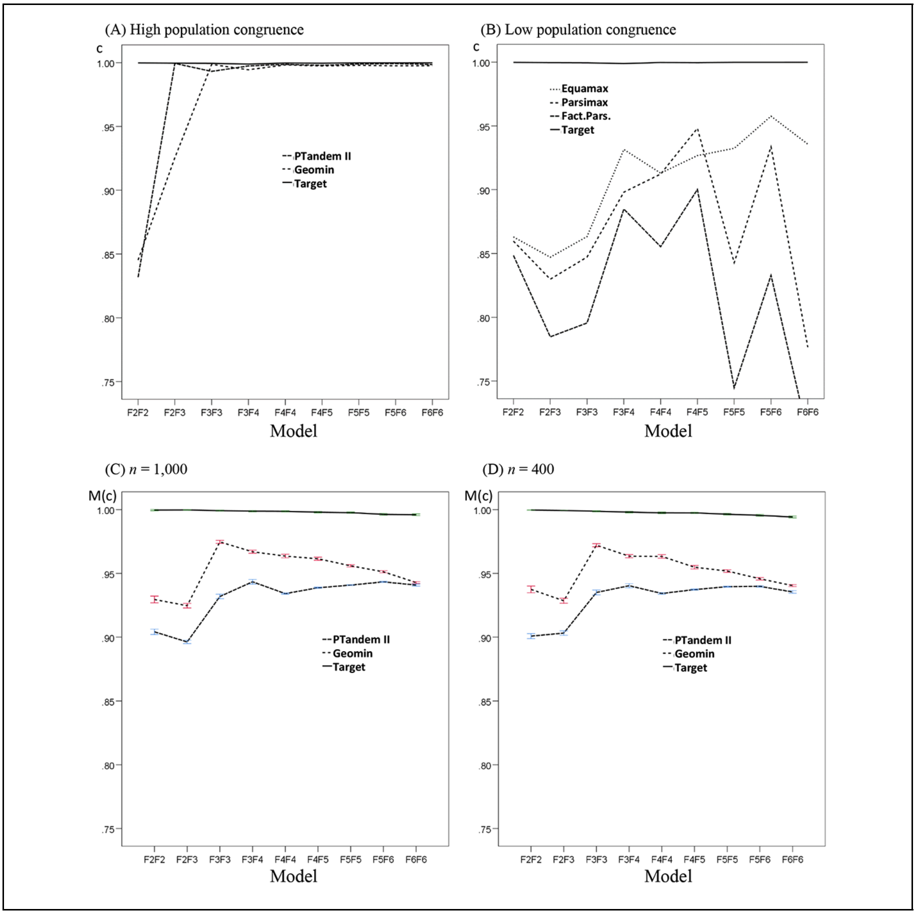

Figure 6. Population congruence (c) in (A) and (B) and sample average of *M*(c) in (C) and (D) for oblique loading matrices with oblique population two-facet simple structure for l = .60, factor-intercorrelation r_f = .30; the model axis gives the number of factors for each of the two facets; the error bars mark the 95% confidence interval. "PTandem II" denotes orthogonal Tandem II pre-rotation followed by Promax (Power = 2) rotation.

loadings with a pattern of ideal BIS loadings was .85. All BIS factors, with exception of the figural intelligence factor (F) were identified. Although the congruence of the Promax-based Tandem II target loading pattern with the ideal BIS loadings was only slightly larger (.86), the combination of salient and nonsalient loadings corresponds to the expected BIS loading pattern (see Table 3) and the figural intelligence factor is found. Thus, the four operation factors (S, M, C, R) and the three content factors (F, V, N) of the BIS were clearly found in the Promax-based Tandem II solution.

Table 3. Promax-Based Tandem II and Tandem II Rotated Factor Loadings of 12 Cross-Product Task Aggregates of the BIS-4 Test.

	Promax-based Tandem II (oblique)Congruence = .86; RMSE = .19							Tandem II (orthogonal)Congruence = .85; RMSE = .21						
Structure	S	M	C	R	F	V	N	S	M	C	R	?	V	N
SF	**.34**	.16	.10	.04	**.75**	−.27	.11	**.45**	.14	.09	−.01	−.01	.22	.14
SV	**.83**	−.09	−.07	−.04	.29	**.53**	.04	.29	.08	.04	.04	**.32**	**.64**	.13
SN	.11	.02	.04	.04	.21	.01	**.85**	.29	.19	.02	.15	.14	.15	**.57**
MF	−.01	**.75**	.10	.24	**.42**	.00	−.16	.25	**.30**	.10	.09	.07	.07	.04
MV	**.40**	**.64**	−.09	−.10	.06	**.53**	−.03	.16	**.36**	.02	−.02	.27	**.34**	.10
MN	−.07	**.91**	−.03	−.10	.16	.05	.26	.23	**.49**	.02	−.04	.11	.05	.25
CF	−.07	−.02	**.90**	−.11	.21	.23	.14	.24	.09	**.65**	−.02	.19	.07	.14
CV	.22	.07	**.55**	.02	−.08	**.71**	−.15	.05	.09	**.41**	.09	**.33**	.25	.00
CN	−.06	−.03	**.46**	.05	.19	.07	**.46**	.18	.09	.25	.09	.11	.04	.26
RF	.00	−.04	−.08	**.86**	.18	.16	.03	.13	−.01	.00	**.59**	.15	.07	.15
RV	**.31**	−.05	.13	.24	−.24	**.79**	.13	−.02	.06	.15	.27	**.37**	.29	.16
RN	−.08	.13	.00	**.51**	−.09	.33	**.43**	.06	.14	.03	**.44**	.24	.06	**.35**
Factor intercorrelations														
S	1.00													
M	.16	1.00												
C	−.08	.06	1.00											
R	−.08	.00	−.05	1.00										
F	.14	.07	.17	.06	1.00									
V	.15	−.01	.18	.19	.07	1.00								
N	−.01	.14	−.08	.11	.17	.14	1.00							

Note. BIS = Berlin Model of Intelligence Structure; RMSE = root mean squared error; S = processing speed; M = memory; C = creativity; R = reasoning/processing capacity; F = figural intelligence; V = verbal intelligence; N = numerical intelligence. A gray background is given where salient loadings are expected according to the BIS; loadings $\geq$ .30 are given in boldface.

The factor inter-correlations are rather small, which could be due to the faceted structure, but also to the rather age-homogeneous sample.

Discussion

It was investigated by means of a simulation study how well loading patterns with a two-facet simple structure can be identified by means of rotation methods used in EFA. For population models based on uncorrelated factors orthogonal Target rotation, Varimax, Equamax, Parsimax, Factor Parsimony, Tandem I, Tandem II, Infomax, and McCammon's rotation were investigated. For population models based on correlated factors oblique Target rotation, oblique versions of Equamax, Parsimax, Factor Parsimony, Oblimin, Geomin, and a Promax-based Tandem II rotation were investigated. The congruence of the sample loading pattern with corresponding population two-facet loading patterns and the RMSE between sample loading pattern and population loading pattern were investigated.

When $q-1$ factors were extracted and q factors were rotated by means of a four-step procedure described in the introduction, the results for uncorrelated factors were as follows: With an increasing number of factors, Equamax, Parsimax, Factor Parsimony, and Tandem II rotation of population data resulted in a nearly perfect representation of two-facet simple structures. In line with these results, orthogonal Target rotation of sample data resulted in mean congruences greater than .98 with the two-facet population models when the salient loadings were greater than .50. When the salient loadings were .40 mean congruences greater than .98 only occurred with n = 1,000. To sum up, orthogonal two-facet loading patterns can be identified by means of EFA when the salient loadings are greater than .50 or with very large sample sizes. For the population models based on correlated factors only Geomin and Promax-based Tandem II rotation of population data resulted in a nearly perfect representation of two-facet simple structures. Therefore, only these two methods were investigated in the simulation study based on samples. Even with salient loadings of .60 and n = 1,000 cases only a mean congruence of .95 was obtained. This indicates that the identification of oblique two-facet simple structures is possible but more difficult than the identification orthogonal two-facet simple structures. Overall, the results imply that large sample sizes are needed for the identification of two-facet loading patterns by means of EFA and subsequent factor rotation.

In an empirical example based on data from an intelligence test, the two-facet BIS model was identified on the level of task aggregates corresponding to the cross-products of the BIS factors by means of EFA with subsequent Promax-based Tandem II rotation. This result is remarkable because only the four BIS operation factors have previously been found in EFA of the 12 task aggregates corresponding to the cross-products (Bucik & Neubauer, 1996). However, Bucik and Neubauer (1996) used Varimax rotation for which the simulation study reveals that it is not suitable for the identification of faceted loading patterns. Moreover, the comparison of the oblique Promax-based Tandem II solution with the orthogonal Tandem II solution reveals that the identification of an empirical two-facet model can be facilitated by means of oblique rotation. This result was not obtained in the simulation studies because in these studies orthogonal rotation methods were performed for orthogonal two-facet population models, whereas here, the orthogonal Tandem II rotation was performed for the slightly oblique two-facet BIS data. The strategy to perform an orthogonal Tandem II rotation followed by a Promax-based Tandem II oblique rotation could help find a two-facet solution when it is a priori unknown whether an orthogonal or oblique model is most appropriate for a given data set.

Limitations

A main limitation of the present study is due to the search space of the simulation study. Aspects that are not covered by the present simulation study are the systematic investigation of the effect of the number of variables per factor (which was rather small in the present study) and the effect of variable population main loadings. Of course, a further limitation is that three-facet models were not considered. However, the results for the two-facet models indicate that it would be even more difficult to identify such complex models by means of factor rotation.

Conclusion

It is concluded that orthogonal two-facet models can best be identified by means of EFA with subsequent Infomax, McCammon rotation for a small number of factors and with subsequent Tandem II rotation for a larger number of factors, but that a rather high data quality is necessary. Oblique two-facet models can be identified by means of Geomin and Promax-based Tandem II rotation. Although Geomin rotation was slightly superior to Promax-based Tandem II rotation, the empirical example shows that the combination of the orthogonal Tandem II rotation with the corresponding Promax-based Tandem II rotation may reveal the advantages of obliqueness without alternation of the basic rotational criterion. Therefore, we recommend the combination of Tandem II and Promax-based Tandem II rotation when it is unknown whether a model should be oblique. When an oblique structure can clearly be expected Geomin rotation should be performed. In any case, the sample size should be large and the main factor loadings should be about .50. However, besides the requirements of high data quality there are no fundamental barriers to the identification of two-facet simple structures by means of EFA with subsequent factor rotation.

Acknowledgments

This work has been stimulated by Adolf Otto Jäger who was a great expert in faceted models of intelligence and who would have celebrated his hundredth birthday on 25th June 2020.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

ORCID iD

André Beauducel https://orcid.org/0000-0002-9197-653X

Supplemental Material

Supplemental material for this article is available online.

References

Asparouhov, T., & Muthén, B. (2009). Exploratory structural equation modeling. *Structural Equation Modeling*, *16*(3), 397-438. https://doi.org/10.1080/10705510903008204

Beauducel, A., & Kersting, M. (2002). Fluid and crystallized intelligence and the Berlin Model of Intelligence Structure (BIS). *European Journal of Psychological Assessment*, *18*(2), 97-112. https://doi.org/10.1027//1015-5759.18.2.97

Bernaards, C. A., & Jennrich, R. I. (2005). Gradient projection algorithms and software for arbitrary rotation criteria in factor analysis. *Educational and Psychological Measurement*, *65*(5), 676-696. https://doi.org/10.1177/0013164404272507

Box, G. E. P., & Muller, M. E. (1958). A note on the generation of random normal deviates. *Annals of Mathematical Statistics*, *29*(2), 610-611. https://doi.org/10.1214/aoms/1177706645

Browne, M. W. (2001). An overview of analytic rotation in exploratory factor analysis. *Multivariate Behavioral Research*, *36*(1), 111-150. https://doi.org/10.1207/S15327906MBR3601_05

Bucik, V., & Neubauer, A. (1996). Bimodality in the Berlin Model of Intelligence Structure (BIS): A replication study. *Personality and Individual Differences*, *21*(6), 987-1005. https://doi.org/10.1016/S0191-8869(96)00129-8

Campbell, D. T., & Fiske, D. W. (1959). Convergent and discriminant validation by the multitrait-multimethod matrix. *Psychological Bulletin*, *56*(2), 81-105. https://doi.org/10.1037/h0046016

Comrey, A. L. (1967). Tandem criteria for analytic rotation in factor analysis. *Psychometrika*, *32*(2), 277-295. https://doi.org/10.1007/BF02289422

Crawford, C. B., & Ferguson, G. A. (1970). A general rotation criterion and its use in orthogonal rotation. *Psychometrika*, *35*(3), 321-332. https://doi.org/10.1007/BF02310792

Eid, M. (2000). A multitrait–multimethod model with minimal assumptions. *Psychometrika*, *65*(2), 241-261. https://doi.org/10.1007/BF02294377

Eid, M., Lischetzke, T., Nussbeck, F. W., & Trierweiler, L. I. (2003). Separating trait effects from trait-specific method effects in multitrait-multimethod models: A multiple indicator

CTC(M–1) model. *Psychological Methods*, *8*(1), 38-60. https://doi.org/10.1037/1082-989X.8.1.38
Grayson, D., & Marsh, H. W. (1994). Identification with deficient rank loading matrices in confirmatory factor analysis: Multitrait-multimethod models. *Psychometrika*, *59*(1), 121-134. https://doi.org/10.1007/BF02294271
Guilford, J. P. (1967). *The nature of human intelligence*. McGraw-Hill.
Guilford, J. P. (1975). Factors and factors of personality. *Psychological Bulletin*, *82*(5), 802-814. https://doi.org/10.1037/h0077101
Guilford, J. P. (1988). Some changes in the structure-of-intellect model. *Educational and Psychological Measurement*, *48*(1), 1-4. https://doi.org/10.1177/001316448804800102
Guttman, L. (1954). An outline of some new methodology for social research. *Public Opinion Quarterly*, *18*(4), 395-404. https://doi.org/10.1086/266532
Guttman, L., & Levy, S. (1991). Two structural laws for intelligence tests. *Intelligence*, *15*(1), 79-103. https://doi.org/10.1016/0160-2896(91)90023-7
Hendrickson, A. E., & White, P. O. (1964). Promax: A quick method for rotation to oblique simple structure. *British Journal of Statistical Psychology*, *17*(1), 65-70. https://doi.org/10.1111/j.2044-8317.1964.tb00244.x
Holzinger, K. J., & Swineford, F. (1937). The bi-factor method. *Psychometrika*, *2*(1), 41-54. https://doi.org/10.1007/BF02287965
Jäger, A. O. (1982). *Mehrmodale klassifikation von intelligenzleistungen: Experimentell kontrollierte weiterentwicklung eines deskriptiven intelligenzstrukturmodells* [Multimodel classification of intelligence performance: Experimentally controlled development of a descriptive model of intelligence structure]. *Diagnostica*, *23*, 195-225.
Jäger, A. O. (1984). Intelligenzstrukturforschung: Konkurrierende modelle, neue entwicklungen, perspektiven [Structural research on intelligence: Competing models, new developments, perspectives]. *Psychologische Rundschau*, *35*(1), 21-35.
Jäger, A. O., Süß, H. M., & Beauducel, A. (1997). *Berliner Intelligenzstruktur-Test. BIS-Test, Form 4* [Test for the Berlin model of intelligence structure, Form 4]. Hogrefe.
Jennrich, R. I. (2001). A simple general procedure for orthogonal rotation. *Psychometrika*, *66*(2), 289-306. https://doi.org/10.1007/BF02294840
Jennrich, R. I., & Bentler, P. M. (2011). Exploratory bi-factor analysis. *Psychometrika*, *76*(4), 537-549. https://doi.org/10.1007/S11336-011-9218-4
Jennrich, R. I., & Sampson, P. F. (1966). Rotation for simple loadings. *Psychometrika*, *31*(3), 313-323. https://doi.org/10.1007/BF02289465
Kaiser, H. F. (1958). The varimax criterion for analytic rotation in factor analysis. *Psychometrika*, *23*(3), 187-200. https://doi.org/10.1007/BF02289233
MacCallum, R. C., Roznowski, M., & Necowitz, L. B. (1992). Model modifications in covariance structure analysis: The problem of capitalizing on chance. *Psychological Bulletin*, *111*(3), 490-504. https://doi.org/10.1037/0033-2909.111.3.490
Marsh, H. W., & Bailey, M. (1991). Confirmatory factor analyses of multitrait-multimethod data: A comparison of alternative models. *Applied Psychological Measurement*, *15*(1), 47-70. https://doi.org/10.1177/014662169101500106
McCammon, R. B. (1966). Principal component analysis and its application in large-scale correlation studies. *Journal of Geology*, *74*(5), 721-733. https://doi.org/10.1086/627207
McKeon, J. J. (1968). *Rotation for maximum association between factors and tests* [Unpublished manuscript]. George Washington University.

Mulaik, S. A., & Quartetti, D. A. (1997). First order or higher order general factor? *Structural Equation Modeling*, *4*(3), 193-211. https://doi.org/10.1080/10705519709540071.

Pfister, H.-R., & Jäger, A. O. (1992). *Topografische Analysen zum Berliner Intelligenstrukturmodell* [Topographic analyses of the Berlin Intelligence Structure (BIS) model]. *Diagnostica*, *38*, 91-116.

Reise, S. P. (2012). The rediscovery of bifactor measurement models. *Multivariate Behavioral Research*, *47*(5), 667-696. https://doi.org/10.1080/00273171.2012.715555

Saunders, D. R. (1962). *Trans-varimax: Some properties of the Ratiomax and Equamax criteria for blind orthogonal rotation* [Paper presentation]. American Psychological Association, St. Louis, MO, USA.

Schlesinger, I. M., & Guttman, L. (1969). Smallest space analysis of intelligence and achievement tests. *Psychological Bulletin*, *71*(2), 95-100. https://doi.org/10.1037/h0026868

Schmid, J., & Leiman, J. M. (1957). The development of hierarchical factor solutions. *Psychometrika*, *22*(1), 53-61. https://doi.org/10.1007/BF02289209

Schmitt, T. A., & Sass, D. A. (2011). Rotation criteria and hypothesis testing for exploratory factor analysis: Implications for factor pattern loadings and interfactor correlations. *Educational and Psychological Measurement*, *71*(1), 95-113. https://doi.org/10.1177/0013164410387348

Schönemann, P. H. (1966). A generalized solution of the orthogonal Procrustes problem. *Psychometrika*, *31*(1), 1-10. https://doi.org/10.1007/BF02289451

Shye, S. (1998). Modern facet theory: Content design and measurement in behavioral research. *European Journal of Psychological Assessment*, *14*(2), 160-171. https://doi.org/10.1027/1015-5759.14.2.160

Süß, H.-M., & Beauducel, A. (2005). Faceted models of intelligence. In O. Wilhelm & R. Engle (Eds.). *Understanding and measuring intelligence* (pp. 313-332). Sage.

Süß, H.-M., & Beauducel, A. (2015). Modeling the construct validity of the Berlin Intelligence Structure Model. *Estudos de Psychologia/Pychological Studies*, *32*(1), 13-25. https://doi.org/10.1590/0103-166X2015000100002

Süß, H. M., Oberauer, K., Wittmann, W. W., Wilhelm, O., & Schulze, R. (2002). Working-memory capacity explains reasoning ability-and a little bit more. *Intelligence*, *30*(3), 261-288. https://doi.org/10.1016/S0160-2896(01)00100-3

Thurstone, L. L. (1947). *Multiple factor analysis: A development and expansion of vectors of the mind*. University of Chicago Press.

Tucker, L. R. (1951). A method for synthesis of factor analysis studies (Personnel Research Section Report No. 984). Department of the Army.

Velicer, A. F., & Jackson, D. N. (1990). Component analysis versus common factor analysis: Some issues in selecting an appropriate procedure. *Multivariate Behavioral Research*, *25*(1), 1-28. https://doi.org/10.1207/s15327906mbr2501_1

Weide, A. C., & Beauducel, A. (2019). Varimax rotation based on gradient projection is a feasible alternative to SPSS. *Frontiers in Psychology. Quantitative Psychology and Measurement*, *10*, Article 645. https://doi.org/10.3389/fpsyg.2019.00645

Yates, A. (1987). *Multivariate exploratory data analysis: A perspective on exploratory factor analysis*. State University of New York Press.